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EXTENSION OF NEW MODEL WITH GOOD APPROXIMATION BASED ON RECURRENCE RELATION

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Abstract: By using novel recurrence relation, some models with good approximation are constructed, which include three aspects: extension of curve of degree n, extension of surface of degree n over rectangular domain. First, based on a novel recurrence relation, we define a Quasi-Bernstein-basis of degree n with multiple parameters, which includes the classical Bernstein basis of degree n as a special case and has similar properties with the classic Bernstein basis. And the definition, properties, comer cutting algorithm, adjustable effect and quantification of approximation of related curves are discussed in detail. Next, tased on Quasi-Bernstein-basis, we develop a tensor product surface with multiple parameters over rectangular domain, and discuss continuity of Quasi-Bézier-surface at length. Compared with the existing methods, the proposed models keep can beautiful properties of classical method and the multiple parameters introduced in these models can flexibly adjust shape of the generated model and possess good approximation.
Keywords: Recurrence relation; Bernstein basis; Corner cutting algorithm; De Cateljau-type algotithm

1 INTRODUCTION

For many years, free-form curves and surfaces have been the focus of researchers , The classical Bézier model has been widely used for its simple definition and beautiful properties [1]. But there exist some problems. The mainly weak point is that the generated curves and surfaces are fixed relative to their control polygon. We can only adjust the shape by modifying control points [2]. The rational curve and surface can address above problems. However, there are some drawbacks in rational curves and surfaces as well, such as improper use of weight factors may lead to very bad parameterization, or even destroy the subsequent structure of curve and surface [3]. Thus, the construction of free-form model has been a hot topic.
To retain the strong points of the traditional Bézier method,increase the adjustable flexibility of the generated shape of model and improve its approximation, a large number of scholars have started the study of the Bézier curves and patches with parameters and presented many methods[4-13]. Where the most simplest method is to introduce a single parameter or a polynomial containing a single parameter into polynomial space, trigonometric space and exponential space, etc [4-7]. Nevertheless, the ways mentioned above mainly focus on incorporating one parameters into curves and surfaces, which indicates that the generated curves and surfaces only can integrally move to a designated direction. What meaning is that the local shape of these models can not be changed freely.
In order to enhance the adjusting flexibility of generating curves and surfaces, scholars stated to introduce multiple parameters into the constructed model [4-13]. Hu et al. gave a basis with n+1 shape parameters [6], they lately extended this basis to rectangular domain and discussed properties of adjustment, connection of the related surfaces [8]. However, local parameters and global parameter are not independent. Thus, some common strong points in Ref. 4-15 could be concluded: (a) they discuss the essential properties similar to the classical Bernstein basis ; (b) all curves and surfaces contain multiple parameters, which can be used to locally or globally modify the shape of the generated curves and surfaces [4-13].
However, the above methods introduced multiple parameters by not increasing the degree of curves and surfaces. In other words, the approaching ability of the generated curves and surfaces is limited. Therefore, in order to increase the approaching ability of this kind of models, we only accomplish it by increasing degree of models. But the newly coming problem is that when facing the high-order curve, the range of these different parametric values could be taken is different. Therefore, when using these models to practical design, it is necessary to evaluate the range of different parameters. In view of the above, scholars tried to apply the variable degree as adjusting parameter. Costannini constructed a cubic variable-degree basis functions [14], and discussed the form of B-spline lately. Then, Costannini et al. extended the basis functions to degree n [15]. Zhu et al. gave a class of basis with exponential parameters and extended the basis to triangular domain [9,16]. Although these methods can improve the drawback of approaching ability, they still have some problems, such as complex calculation, complex splicing and so on. Moreover, the key point is that these models could only add two parameters at most, so these parameters only serve as global parameters in modeling design, and lack of locally adjustable ability.
In response to the above questions, the purpose of this paper is to propose a novel extended basis of degree n with multiple parameters, which can infinitely approach control polygon. Firstly, a new Quasi-Bernstein-basis of degree n with 2[n/2] parameters, which includes the classical Bernstein basis of degree n as a special case, is defined. Then, we

use method of tensor product to extend it to rectangular domain and construct a class of Quasi-Bernstein-surfaces of degree n. Compared with the existing method , the proposed method has multiple parameters, which indicates that the generated curves and patches in this paper can approach the control polygon and mesh infinitely. By adjusting the parametric values, the local shape of the generated curves and patches can be predicted.
Others work of this paper can be concluded as follows: Section 2 gives some work which have been done.Section 3 gives definition and properties of Quasi-Bernstein-basis of degree n. Sections 4 discusses the definition,properties, corner cutting algorithm,approaching analysis and comparison and $C^k, C^k(k=0,1,2)$ continuous conditions of the related curve of Quasi-Bernstein-basis of degree n. Section 5 gives the definition, properties and $C^k, C^k(k=0,1,2)$ continuous conditions of Quasi-Bézier-surface of degree n.

2 PRELIMINARIES

Definition1 The definition of classical Bézier curve of order n is:

$$Q(u,v)=\sum_{i=0}^{n} B_{i,j}^{n}(u,v)P_{i,j}^{n}, i,j \in N, i+j=n; u,v \geq 0, u+v=1. \tag{1}$$

Where

$$B_{i,j}^{n}(u,v)=C_{n}^{i}u^{i}v^{j}=\frac{n!}{i!\,j!}u^{i}v^{j},\quad i,j \in N, i+j=n; u,v \geq 0, u+v=1. \tag{2}$$

is the classical Bernstein basis of order n, and $P_{i,j}^{n}$ are control points.

Definition2 The definition of $m \times n$-order classical surfaces over rectangular domain is:

$$Q(u,v;u*,v*)=\sum_{i=0}^{m}\sum_{k=0}^{n} B_{i,j}^{m}(u,v)B_{k,l}^{n}(u*,v*)P_{i,k} \tag{3}$$

where $B_{i,j}^{n}(u,v)$ and $B_{k,l}^{n}(u*,v*)$ are the Bernstein basis given in (2).

Definition 3 For $i,j,k \in N, i+j+k=n, n \geq 2$ and $D=\{(u,v,w) | u+v+w=1; u \geq 0, v \geq 0, w \geq 0\}$ the classical n-order Bernstein-Bézier patches over triangular domain are defined as follows:

$$Q(u,v,w)=\sum_{i+j+k=n} B_{i,j,k}^{n}(u,v,w)P_{i,j,k}^{n} \tag{4}$$

where $$B_{i,j,k}^{n}(u,v,w)=\frac{n!}{i!\,j!\,k!}u^{i}v^{j}w^{k},(u,v,w)\in D \tag{5}$$

are classical Bernstein-Bézier basis, and $P_{i,j}^{n}$ are control points.

3 QUASI-BERNSTEIN-BASIS OF DEGREE *N*

3.1 Construction of Quasi-Bernstein-Basis

In this subsection, by using a novel recursive relation, the definition of new basis of degree n with multiple parameters will be given.

Definition 4 For $i,j \in N, i+j=n, n \geq 2$ and $d=\{(u,v) | u+v=1, u \geq 0, v \geq 0\}$,the Quasi-Bernstein-basis (QB-basis for short) of degree n over the domain d , with 2[n/2] parameters

$\alpha_{x,y}^{n-1}(x,y \in N, x+y=n-1)$ are defined by

$$A_{i,j}^{n}(u,v)=f_{i-1,j}^{n-1}(u,v)B_{i-1}^{n-1}(u,v)+g_{i,j-1}^{n-1}(u,v)B_{i,j-1}^{n-1}(u,v) \tag{6}$$

where

$$B_{x,y}^{n-1}(u,v)=\frac{(n-1)!}{x!\,y!}u^{x}v^{y},(x,y \in N, x+y=n-1) \tag{7}$$

are Bernstein basis of degree n-1 over the domain d, and

$$\begin{pmatrix} f_{i-1,j}^{n-1}(u,v)=\begin{cases}\dfrac{1}{1+\alpha_{i-1,j}^{n-1}(1-u)}u, & i-[(n+1)/2]>0,\\ 1-\dfrac{1}{1+\alpha_{i-1,j}^{n-1}(1-v)}v, & i-[(n+1)/2]\le 0\end{cases}\\ g_{i,j-1}^{n-1}(u,v)=\begin{cases}\dfrac{1}{1+\alpha_{i,j-1}^{n-1}(1-v)}v, & i-[(n+1)/2]>0,\\ 1-\dfrac{1}{1+\alpha_{i,j-1}^{n-1}(1-u)}u, & i-[(n+1)/2]\le 0\end{cases}\end{pmatrix} \tag{8}$$

if n is even, and

$$\begin{pmatrix} f_{i-1,j}^{n-1}(u,v)=\begin{cases}\dfrac{1}{1+\alpha_{i-1,j}^{n-1}(1-u)}u, & i-[(n+1)/2]>0,\\ & i-1,\ j\ne[n/2],\\ 1-\dfrac{1}{1+\alpha_{i-1,j}^{n-1}(1-v)}v, & i-[(n+1)/2]\le 0\\ & i-1,\ j\ne[n/2],\\ u, & others,\end{cases}\\ g_{i,j-1}^{n-1}(u,v)=\begin{cases}\dfrac{1}{1+\alpha_{i-1,j}^{n-1}(1-v)}v, & j-[(n+1)/2]>0,\\ & i,\ j-1\ne[n/2],\\ 1-\dfrac{1}{1+\alpha_{i-1,j}^{n-1}(1-u)}u, & j-[(n+1)/2]\le 0\\ & i,\ j-1\ne[n/2],\\ v, & others,\end{cases}\end{pmatrix} \tag{9}$$

If n is odd, and $[n/2]\,[n/2]=\begin{cases} n/2, if\ n\ is\ even,\\ (n-1)/2, if\ n\ is\ odd.\end{cases}$

Moreover, we always set $\alpha_{x,y}^{n-1}=0$ and $B_{x,y}^{n-1}=0$ whenever one subscript is equal to -1.

Remark 1 Obviously, the QB-basis proposed in this paper has similar form and properties to the classic Bernstein basis of degree n. The difference is that the QB-basis has 2[n/2] parameters which can improve the shortcomings of the classic Bernstein basis. In particular, if set all parameters $\alpha_{x,y}^{n-1}=0$, the QB-basis of degree n will degenerate to formula (2).

Remark 2 The QB-basis defined in (6) possesses 2[n/2] parameters. Actually, the parameters serve as local parameter, meaning that there exist one single curve segment will be changed when altering a any parameter. In particular, the parameters of QB-basis serve as global parameter when the whole parameters $\alpha_{x,y}^{n-1}$ are taken a same value.

3.2 Properties of the QB-Basis

Theorem 1 The properties of QB-basis given in (6) can be obtained as follows:
(1)Partition of unity: $\sum_{i+j=n} A_{i,j}^{n}(u,v)=1$.

(2)Non-negativity: If all parameters satisfy $\alpha_{x,y}^{n-1}\ge 0$, then $A_{i,j}^{n}(u,v)\ge 0$.

(3)Symmetry: If $\alpha_{x,y}^{n-1}=\alpha_{y,x}^{n-1}$, we have $A_{i,j}^{n}(u,v)=A_{j,i}^{n}(v,u)$, see Figure 1($a$) and (1)($b$) .

(4)Terminal properties:When $n \geq 3$, for v = 1-u and $i, j \in$ N, $j = n - i$,it follows that

$$\begin{cases} A_{n,0}^{n}(1,0)=1, A_{i,n-i}^{n}(1,0)=0(i\neq n), \\ A_{0,n}^{n}(0,1)=1, A_{i,n-i}^{n}(0,1)=0(i\neq 0). \end{cases} \tag{10}$$

By directly differentialing concerning u, it follows that

$$\left[A_{i,n-i}^{n}(1,0)\right]'=\begin{cases} n+\alpha_{n-1,0}^{n-1}, i=n, \\ -(n+\alpha_{n-1,0}^{n-1}), i=n-1 \\ 0, others, \end{cases} \tag{11}$$

$$\left[A_{i,n-i}^{n}(0,1)\right]'=\begin{cases} -(n+\alpha_{0,n-1}^{n-1}), i=0, \\ n+\alpha_{0,n-1}^{n-1}, \quad i=1 \\ 0, \quad others, \end{cases} \tag{12}$$

By 2-th differentialing with regard to u, it follows that

$$\left[A_{i,n-i}^{3}(1,0)\right]''=\begin{cases} 2(\alpha_{2,0}^{2})^{2}-6\alpha_{2,0}^{2}+6, \quad i=3, \\ -2(\alpha_{2,0}^{2})^{2}+6\alpha_{2,0}^{2}-12, i=2 \\ 6, \quad i=1 \\ 0, \quad others, \end{cases} \tag{13}$$

$$\left[A_{i,n-i}^{3}(0,1)\right]''=\begin{cases} 2(\alpha_{0,2}^{2})^{2}-6\alpha_{0,2}^{2}+6, \quad i=0, \\ -2(\alpha_{0,2}^{2})^{2}+6\alpha_{0,2}^{2}-12, i=1, \\ 6, \quad i=2, \\ 0, \quad others, \end{cases} \tag{14}$$

if n = 3, and

$$\left[A_{i,n-i}^{n}(1,0)\right]''=\begin{cases} 2\alpha_{n-1,0}^{n-1}(\alpha_{n-1,0}^{n-1}+n)+n(n-1), \quad i=n, \\ -2\alpha_{n-1,0}^{n-1}(\alpha_{n-1,0}^{n-1}+n)-(n-1)(n+2\alpha_{n-2,1}^{n-1}), i=n-1 \\ 2(n-1)\alpha_{n-2,1}^{n-1}, \quad i=n-2 \\ 0, \quad others, \end{cases} \tag{15}$$

$$\left[A_{i,n-i}^{n}(0,1)\right]''=\begin{cases} 2\alpha_{0,n-1}^{n-1}(\alpha_{0,n-1}^{n-1}+n)+n(n-1), \quad i=0, \\ -2\alpha_{0,n-1}^{n-1}(\alpha_{0,n-1}^{n-1}+n)-(n-1)(n+2\alpha_{1,n-2}^{n-1}), i=1 \\ 2(n-1)\alpha_{1,n-2}^{n-1}, \quad i=2 \\ 0, \quad others, \end{cases} \tag{16}$$

if n > 3.

(5)Totally positive property: If all $\alpha_{x,y}^{n-1}>0, A_{i,j}^{n}(u,v)$, form a class of totally positive basis.

(6)Linear independence: If all $\alpha_{x,y}^{n-1}>0,$ the QB-basis given in (6) are linear independence.

4 QUASI-BÉZIER-CURVES OF DEGREE N

4.1 Definition of Quasi-Bézier-Curves

Definition 5 Given control points $P_{i,j}\in R^{3}(i,j\in N, i+j=n, n\geq 2),$ and a domain $d=\{(u,v)\,|\,u+v=1, u\geq 0, v\geq 0\}$. We call

$$R^n(u,v)=\sum_{i+j=n}P_{i,j}A_{i,j}^n(u,v),(u,v)\in d, \tag{17}$$

the Quasi-Bézier-curves (QB-curves for short) of degree n with $2[n/2]$ parameters, where $A_{i,j}^n(u,v)$ are the QB-basis given in (6).

Theorem 2 From Theorem 1, some significant properties of the QB-curves will be organized:

(1)End-points properties: When $n \geq 3$, for $v = 1 — u$ and $i, j \in N, j = n — i$, it follows that

$$\begin{cases} R^n(1,0)=P_{n,0}, \\ R^n(0,1)=P_{0,n}. \end{cases} \tag{18}$$

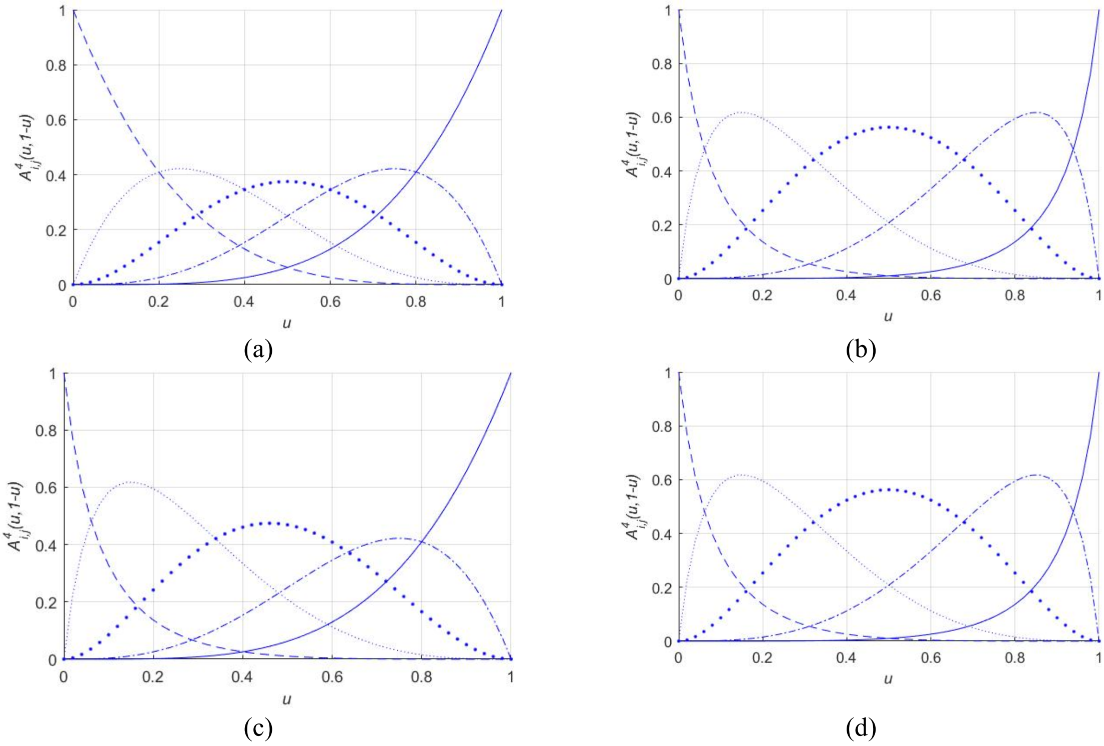

Figure 1 Image of QB-basis with Different Parameters

where, when is shown in Figure 1(a), when is shown in Figure 1(b), when is shown in Figure 1(c), and when is shown in Figure 1(d).

By directly differentialing concerning , it follows that

$$\begin{cases} [R^n(1,0)]'=(n+\alpha_{n-1,0}^{n-1})(P_{n,0}-P_{n-1,1}), \\ [R^n(0,1)]'=(n+\alpha_{0,n-1}^{n-1})(P_{1,n-1}-P_{0,n}). \end{cases} \tag{19}$$

By 2-th differentialing concerning u, it follows that

$$\begin{cases} [R^3(1,0)]''=[2(\alpha_{2,0}^2)^2-6\alpha_{2,0}^2+6](P_{3,0}-P_{2,1})+6(P_{1,2}-P_{2,1}), \\ [R^3(0,1)]''=[2(\alpha_{0,2}^2)^2-6\alpha_{0,2}^2+6](P_{0,3}-P_{1,2})+6(P_{2,1}-P_{1,2}) \end{cases} \tag{20}$$

if $n = 3$, and

$$\begin{cases} [R^n(1,0)]''=[2\alpha_{n-1,0}^{n-1}(\alpha_{n-1,0}^{n-1}+n)+n(n-1)(P_{n,0}-P_{n-1,1})+2(n-1)\alpha_{n-2,1}^{n-1}(P_{n-2,2}-P_{n-1,1}), \\ [R^n(0,1)]''=[2\alpha_{0,n-1}^{n-1}(\alpha_{0,n-1}^{n-1}+n)+n(n-1)(P_{0,n}-P_{1,n-1})+2(n-1)\alpha_{1,n-2}^{n-1}(P_{2.n-2}-P_{1,n-1}), \end{cases} \tag{21}$$

if $n > 3$

From formulas (17)-(21), the following conclusions can be deduced:

(1) (a) the QB-curves interpolate to both endpoints of the control polygon, and tangent to the first and the last edge of the control polygon at both endpoints, and (b) the magnitude of the first and the second derivative at both endpoints on QB-curves can be adjusted handily by modifying the shape parameters, which brings considerable convenience to smooth connecting.

(2) Convex hull and affine invariance: Since the QB-basis given in 6 has the properties of partition of unity and non-negativity, then the QB-curves have convex hull and affine invariance.

(3) Symmetry:Given control polygons $P_{n,0}, P_{n-1,1}, \cdots P_{0,n}$ and $P_{0,n}, P_{1,n-1}, \cdots P_{n,0}$, the two class of control polygons define a same QB-curve when $\alpha_{x,y}^{n-1} = \alpha_{y,x}^{n-1}$, that is $R^n(u,v) = R^n(v,u)$.
(4) Variation diminishing property: Since the QB-basis form a class of totally positive basis, then the QB-curves possess variation diminishing property.
(5) Convexity preserving: Since the QB-curves possess variation diminishing property, then the QB curves also possess convexity preserving.
(6) Shape adjustable property: Given a control polygon, the shape of the classical Bézier curve will be completely determined. But for an QB-curve, it is not the case.When fixed the control polygon, the shape of an QB-curve can still be adjusted by changing the $2[n/2]$ parameters.

5 CONCLUSION

This paper proposes a novel class of Quasi-Bernstein basis functions constructed via a new recurrence relation, incorporating multiple shape parameters. On this basis, corresponding Quasi-Bézier curve and surface models are developed. These basis functions not only retain the desirable properties of classical Bernstein bases (such as partition of unity, non-negativity, and symmetry) but also significantly enhance shape adjustability through the introduction of multiple local parameters. The proposed models maintain the advantages of traditional Bézier methods while achieving infinite approximation to the control polygon or mesh via parameter modulation, offering improved local shape control and modeling flexibility.Although the proposed models exhibit excellent performance in shape adjustment and approximation capability, several limitations remain:The number of parameters increases with the degree of the curve/surface, which may complicate parameter selection and optimization in practical applications.The allowable ranges for different parameters in high-degree curves and surfaces may vary, and a unified method for determining these ranges has not been established.The computational complexity of the model is relatively high, which could impact efficiency in real-time interactive design. The current work is limited to tensor-product surfaces over rectangular domains; extension to triangular or other complex domains has not been explored.
Future research will focus on the following aspects:
1. Developing an automatic parameter optimization mechanism to reduce manual intervention and improve modeling efficiency.
2. Extending the proposed basis functions to construct surfaces over triangular domains, further improving their applicability in real-world CAD/CAM systems.
3. Exploring potential applications of the model in isogeometric analysis, reverse engineering, and other related fields.

COMPETING INTERESTS

The authors have no relevant financial or non-financial interests to disclose.

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